Diffraction-free field in a planar nonlinear waveguide

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A vector nonlinear Schrödinger equation and corresponding laws of conservation are derived for both a bulk medium and a nonlinear planar waveguide. After physical interpretation of these laws of conservation the solutions of scalar nonlinear Schrödinger equation are found using a condition for diffraction-free fields.

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I. INTRODUCTION

In recent years, much work has appeared in the area of diffraction-free propagation of light. The effect of diffraction is usually involved with the transverse spreading of a spatially limited beam. With respect to our knowledge, the generally valid quantitative definition of diffraction is not known. Nevertheless, several methods of how to suppress the effect of diffraction are known, although the cause of diffraction is not known.

In vacuum or in a homogeneous isotropic material the diffraction does not exist only for the particular space distribution fields [1–19]. Another possible way to suppress the diffraction is to use a material with an inhomogeneously distributed index of refraction, i.e., linear waveguide [20,21] (guided fields do not diffract).

The diffraction can also be compensated by a nonlinear interaction. A typical case is the interaction with a nonlinear medium having Kerr nonlinearity, i.e., a medium with an intensity-dependent index of refraction $n = n_0 + n_2|\mathbf{E}|^2$, where n_0 is linear part of the index of refraction, n_2 is Kerr constant, and $\mathbf{E}$ is the vector of electric field intensity. The propagation in such a medium could be described (in some approximation) by means of the nonlinear Schrödinger equation (NSE), which controls the behavior of $\mathbf{E}$.

This equation admits different solutions. The spatial optical solitons, which are an example of a solution, are diffraction-free. The solitons may be divided into two categories: scalar and vector. Scalar solitons involve only one polarization component of an optical field [22,23], whereas vector solitons may consist of two polarization components [24].

The character of a scalar soliton and a condition for its existence depend strongly on the dimensionality of the configuration space as well as on the sign of the nonlinearity.

In (1+1)-dimensional [(1+1)D] (one space plus one time dimension) space a soliton exists for either sign of n_2 [22,23]. A bright soliton exists for $n_2 > 0$ and a dark soliton ("kink") exists for $n_2 < 0$.

In two or three dimensions no solitons exist for $n_2 > 0$ [27]. There is a solitary wave solution that is known to be unstable and can collapse to a self-focusing singularity [25]. A soliton solution exists in two dimensions for $n_2 < 0$. These solutions represent vortices in on optical field [26]. Considerably less attention has been devoted to vector spatial solitons [24,28,29].

The general aim of our investigation is to study conditions for the existence of diffraction-free fields and their properties. It is clear from given examples that the problem of diffraction-free fields is very complex. Therefore, we shall confine ourselves in this article to study a diffraction-free field in a Kerr nonlinear medium. There are two main aims of this article. The first aim is to formulate the vector NSE and the corresponding laws of conservation. The second aim is to use a condition for diffraction-free fields to find diffraction-free fields and to give some of their properties.

In Sec. II we shall state the definition of a diffraction-free field [30]. In Sec. III we shall formulate the NSE in vectorial form and the corresponding laws of conservation for the energy density and momentum density of the electric field $\mathbf{E}$. The formulation is given for (3+1)D space (bulk material). The reformulation for (2+1)D (nonlinear planar waveguide) and (1+1)D (paraxial approximation of the previous case) space is given in Sec. IV. In Sec. V the scalar NSE is solved using the condition for the diffraction-free fields. The recently discovered features of known solutions are given. The whole description is formulated for optical pulses (in a quasimonochromatic approximation).

II. DIFFRACTION

The phenomenon of diffraction is well known in optics. Nevertheless, it is very difficult to give an exact and complete definition of diffraction.

One of the phenomena, which demonstrates diffraction during the propagation in a free space, is a transverse spreading of a generally distributed field. A nonzero flow of the energy in the transverse direction corresponds to this phenomenon. As we deal with problems of propagation, we shall apprehend the diffraction as a nonzero flow of energy in transverse direction.

The flow of energy of the electromagnetic field is determined by a Poynting vector $\mathbf{S}$. Let us introduce a vector $\vec{\beta}$, which determines a dominant direction of propagation. The Poynting vector can be decomposed as $\mathbf{S} = \mathbf{S}_L + \mathbf{S}_T$, where $\mathbf{S}_L$ is a longitudinal part, which is parallel to $\vec{\beta}$, and $\mathbf{S}_T$ is transversal part, which is perpendicular to $\vec{\beta}$. The relation $\mathbf{S}_T \cdot \mathbf{S}_L = \mathbf{S}_T \cdot \vec{\beta} = 0$ must hold. The longitudinal part $\mathbf{S}_L$ determines the flow of energy in a dominant ($\vec{\beta}$) direction of propagation of the electromagnetic field. The transversal part $\mathbf{S}_T$ determines the flow of energy perpendicular to this direction.

The subject of our study is a diffraction-free propagation of a field. We shall use the following definition of a diffraction-free field [30]: if in an electromagnetic field the condition

$$\nabla \cdot \mathbf{S}_T = 0, \tag{1}$$

holds, where ∇ is a nabla operator, then this field is diffraction-free. This definition of the diffraction-free field admits nonzero transversal flow (e.g., a vortex with zero radial flow but nonzero tangential flow).

The particular class of the diffraction-free fields fulfilling condition (1) is

$$\mathbf{S}_T = \mathbf{0}, \tag{2}$$

which is a particular case of condition (1). In this article we shall investigate this class of the diffraction-free field.

III. DESCRIPTION

A wave equation for the electric field $\hat{\mathbf{E}}$ is usually used to solve problems of nonlinear optics. If one wants to have complete electromagnetic description of the problem solved, the magnetic field $\hat{\mathbf{H}}$ has to be calculated using the Maxwell equations.

One of the very important quantities in the electromagnetic field that cannot be generally decomposed to the separate contributions of the electric and the magnetic fields is the Poynting vector describing the density of flow (flux) of an electromagnetic energy. A density of flow of the electric field can also be defined on the basis of the wave equation. What the connection between this density of flow of an electric field and the Poynting vector is seems to be natural question. This connection is given in the Appendix.

A. Equation of propagation

As remarked above, problems of nonlinear optics are usually described by means of the wave equation

$$\nabla^2 \hat{\mathbf{E}} - \mu_0 \epsilon \partial_{tt} \hat{\mathbf{E}} = \mu_0 \partial_{tt} \hat{\mathbf{P}}, \tag{3}$$

where ∇ is the nabla operator, $\partial_{tt} \equiv \frac{\partial^2}{\partial t^2}$, ϵ (μ_0) is the permittivity (permeability) of a material that is homogeneous and isotropic, and $\hat{\mathbf{P}}$ is the vector of a nonlinear polarization. The wave equation (3) has been derived from the Maxwell equations under the assumption

$$\nabla \cdot \hat{\mathbf{E}} = -\frac{1}{\epsilon} \nabla \cdot \ \hat{\mathbf{P}} \approx 0. \tag{4}$$

Assuming that a complex vector $\hat{\mathbf{V}}$ has form

$$\hat{\mathbf{V}}(t) = \mathbf{V}(t) e^{-i\omega t}, \tag{5}$$

where ω is the carrier frequency of an electromagnetic field, then using quasimonochromatic approximation for the vectors $\hat{\mathbf{E}}$ and $\hat{\mathbf{P}}$, i.e.,

$$(\partial_t - i\omega)\mathbf{V} \approx -i\omega \mathbf{V}, \tag{6}$$

the wave equation (3) can be rewritten as

$$\nabla^2 \mathbf{E} + 2i\omega\mu_0\epsilon\partial_t \mathbf{E} + \omega^2 \mu_0 \epsilon \mathbf{E} = -\mu_0 \omega^2 \mathbf{P}, \tag{7}$$

where we have kept the first derivative of $\mathbf{E}$ on the left-hand side of this equation.

We shall investigate the propagation of an electric field in a nonlinear Kerr medium. The vector of the nonlinear polarization $\mathbf{P}$ describing the Kerr medium is defined by the relation

$$\mathbf{P} = A|\mathbf{E}|^2 \mathbf{E} + \frac{1}{2} B(\mathbf{E} \cdot \mathbf{E})\mathbf{E}^*, \tag{8}$$

where the asterisk denotes the complex conjugated quantities. The values of the constants A and B depend on the physical mechanism leading to the nonlinear Kerr effect. Substituting (8) into (7) we obtain

$$2i\omega\mu_0\epsilon\partial_t \mathbf{E} + \nabla^2 \mathbf{E} + \omega^2 \mu_0 \epsilon \mathbf{E} = -\mu_0 \omega^2 \left[A|\mathbf{E}|^2 \mathbf{E} + \frac{1}{2} B(\mathbf{E} \cdot \mathbf{E})\mathbf{E}^* \right], \tag{9}$$

which is the vector (3+1)D nonlinear Schrödinger equation. This equation is the starting point for our following considerations.

B. Laws of conservation

Starting from Eq. (9), we arrive at the conservation laws

$$\partial_t w = -\partial_j S_j, \tag{10}$$

$$F_j \equiv \partial_t g_j = \partial_k T_{jk}, \tag{11}$$

where

$$w = \epsilon |\mathbf{E}|^2, \tag{12}$$

$$S_j = -\frac{i}{2\omega\mu_0}(\mathbf{E}^* \cdot \partial_j \mathbf{E} - \mathbf{E} \cdot \partial_j \mathbf{E}^*), \tag{13}$$

$$g_j = \mu_0 \epsilon S_j, \tag{14}$$

$$T_{jk} = -\frac{1}{2\mu_0\omega^2}(\partial_j \mathbf{E} \cdot \partial_k \mathbf{E}^* + \partial_j \mathbf{E}^* \cdot \partial_k \mathbf{E}) + \frac{1}{4\mu_0\omega^2} \partial_{jk} |\mathbf{E}|^2$$

$$+\delta_{jk}\left[A|\mathbf{E}|^4+\frac{1}{2}B|\mathbf{E}^2|^2\right], \tag{15}$$

where $j,k=x,y,z$, δ_{ik} is the Kronecker delta function, and Einstein's summation rule has been used. The quantities introduced have the following physical meaning: w is the energy density of the field (hereafter we shall omit the qualification "electric" because we deal with the electric field only), the vector $\mathbf{S}$ is the average density of flow of the field, the vector $\mathbf{g}$ is a momentum density of the field, the vector $\mathbf{F}$ is a force density of the field, and finally the tensor $\mathbf{T}$ we shall call the stress tensor of the field by analogy with the Maxwell stress tensor. It would be possible to introduce another law of conservation, but ones we have are adequate for our purposes.

Until now we have described propagation in bulk material [i.e., the (3+1)D case]. It is known that the propagation is unstable in this case. The stability of propagation is improved if a planar waveguide structure is introduced into the material.

IV. NONLINEAR PLANAR WAVEGUIDE

We shall study the propagation in a standard nonlinear waveguide which is compounded from a nonlinear thin film surrounded by two linear semispaces with the plane of the film perpendicular to the y direction. Let us decompose the field $\mathbf{E}$ in a form

$$\mathbf{E}(x,y,z,t)=\hat{\mathbf{x}}E_x(x,z,t)u_x(y)+\hat{\mathbf{y}}E_y(x,z,t)u_y(y)+\hat{\mathbf{z}}E_z(x,z,t)u_z(y), \tag{16}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors and $u_j(y)$ are mode functions of the linear waveguide satisfying the equation

$$\partial_{yy}u_j+(k^2-\beta_j^2)u_j=0 \tag{17}$$

and the corresponding boundary conditions for $j=x,y,z$. The quantity k is

$$k^2=\omega^2\mu_0\epsilon=\frac{\omega^2}{v^2}=k_0^2n^2, \tag{18}$$

where the index of refraction n (or ϵ) picks up values describing a structure of waveguide. The quantities β_j are the magnitudes of vectors of propagation in a linear waveguide corresponding to different kinds of modes.

If we suppose that the field propagates in the z direction, then u_x is a mode function of a TE field and u_y and u_z are mode functions of a TM field. Using relations (16) and (17), Eq. (9) can be rewritten in the form

$$2i\omega\mu_0\epsilon\partial_tE_j+\triangle_LE_j+\beta_j^2E_j=-\mu_0\omega^2\Bigg\{A[N_{xj}|E_x|^2+N_{yj}|E_y|^2+N_{zj}|E_z|^2]\frac{1}{N_j}E_j+\frac{1}{2}B[M_{xj}E_x^2+M_{yj}E_y^2+M_{zj}E_z^2]\frac{1}{N_j}E_j^*\Bigg\}, \tag{19}$$

where $\triangle_L\equiv(\partial_{xx}+\partial_{zz})$ and

$$N_j=\int_{-\infty}^{\infty}|u_j|^2\,dy, \tag{20}$$

$$N_{jk}=\int_{-\infty}^{\infty}|u_j|^2|u_k|^2\,dy, \tag{21}$$

$$M_{jk}=\int_{-\infty}^{\infty}u_j^2u_k^{*2}\,dy. \tag{22}$$

Equation (19) describes the behavior of the field in the nonlinear part of the waveguide. It can be simplified for several special cases.

Case (a): The TE field. In this case $u_y=u_z=0$ and Eq. (19) is simplified to

$$2i\omega\mu_0\epsilon\partial_tE_x+\triangle_LE_x+\beta_x^2E_x=-\mu_0\omega^2\frac{N_{xx}}{N_x}\left[A+\frac{1}{2}B\right]|E_x|^2E_x, \tag{23}$$

which is a scalar NSE. $\beta_x\equiv\beta_{\rm TE}$ are the corresponding propagation constants.

Case (b): The TM field. In this case $u_x=0$ and Eq. (19) is simplified to

$$2i\omega\mu_0\epsilon\partial_tE_y+\triangle_LE_y+\beta_y^2E_y=-\mu_0\omega^2\left\{A\left[N_{yy}|E_y|^2+N_{zy}|E_z|^2\right]\frac{1}{N_y}E_y+\frac{1}{2}B\left[M_{yy}E_y^2+M_{zy}E_z^2\right]\frac{1}{N_y}E_y^*\right\}, \tag{24}$$

$$2i\omega\mu_0\epsilon\partial_tE_z+\triangle_LE_z+\beta_z^2E_z=-\mu_0\omega^2\left\{A\left[N_{yz}|E_y|^2+N_{zz}|E_z|^2\right]\frac{1}{N_z}E_z+\frac{1}{2}B\left[M_{yz}E_y^2+M_{zz}E_z^2\right]\frac{1}{N_z}E_z^*\right\}, \tag{25}$$

where $\beta_y\equiv\beta_z\equiv\beta_{\rm TM}$ are corresponding propagation constants. If $u_z\ll u_y$, then neglecting small quantities we obtain

$$2i\omega\mu_0\epsilon\partial_tE_y+\triangle_LE_y+\beta_y^2E_y=-\mu_0\omega^2\frac{N_{yy}}{N_y}\left[A+\frac{1}{2}B\right]|E_y|^2E_y, \tag{26}$$

$$2i\omega\mu_0\epsilon\partial_tE_z+\triangle_LE_z+\beta_z^2E_z=-\mu_0\omega^2\left\{A\frac{N_{yz}}{N_z}|E_y|^2E_z+\frac{1}{2}B\frac{M_{yz}}{N_z}E_y^2E_z^*\right\}. \tag{27}$$

Equation (26) is again the NSE for the field component E_y. Equation (27) is a linear equation for the field component E_z. The behavior of the component E_z depends on the behavior of the component E_y, but not vice versa.

Case (c): Weakly guided field. If the linear refraction index difference is small, then $u_z \ll u_y \approx u_x$ and we have

$$\begin{aligned} N_x &= N_y, \\ N_{xx} &= N_{xy} = N_{yy}, \\ M_{jk} &= N_{jk}. \end{aligned} \tag{28}$$

In the same approximation as in case (b) we obtain

$$2i\omega\mu_0\epsilon\partial_t E_j + \triangle_L E_j + \beta_j^2 E_j = -\mu_0\omega^2 \frac{N_{xx}}{N_x}\left[A|\mathbf{E}|^2 E_j + \frac{1}{2}B\mathbf{E}^2 E_j^*\right], \tag{29}$$

$$\begin{aligned} 2i\omega\mu_0\epsilon\partial_t E_z &+ \triangle_L E_z + \beta_z^2 E_z \\ &= -\mu_0\omega^2 \left\{ A\left[N_{xz}|E_x|^2 + N_{yz}|E_y|^2\right]\frac{1}{N_z}E_z + \frac{1}{2}B\left[M_{xz}E_x^2 + M_{yz}E_y^2\right]\frac{1}{N_z}E_z^* \right\}, \end{aligned} \tag{30}$$

where $j = x, y$, $\beta_x = \beta_{\rm TE}$, $\beta_y \equiv \beta_z = \beta_{\rm TM}$, and

$$\mathbf{E} = \hat{x}E_x + \hat{y}E_y. \tag{31}$$

Equation (29) now represents a coupled NSE. Equation (30) is a linear equation for a longitudinal component of the field. Case (c) includes previous cases.

It is obvious that the influence of the small longitudinal component E_z can be formally described as a perturbation of the field $\mathbf{E}$. The influence of this perturbation has been studied in many papers (see, e.g., [31]), of course, in another context.

Hereafter we shall suppose that a longitudinal component of the field is negligible and therefore we shall confine ourselves to Eq. (29) only. Of course, this assumption is exact only for a TE field.

Paraxial approximation

As we have assumed above, the field $\mathbf{E}$ mainly propagates in the direction $\mathbf{z}$. Then it is convenient to write

$$\mathbf{E} = \mathbf{Q}e^{i\beta z}, \tag{32}$$

where $\beta = \beta_{\rm TE} + \delta\beta = \beta_{\rm TM} - \delta\beta$. If the changes of the field along the $\mathbf{z}$ direction are not too sudden, we can introduce paraxial approximation by means of the relations

$$|\partial_{zz}Q_j| \ll |\beta\partial_z Q_j| \ll |\beta^2 Q_j|, \quad j = x, z. \tag{33}$$

In this approximation Eq. (29) is

$$2i\omega\mu_0\epsilon\partial_t Q_j + 2i\beta\partial z Q_j + \partial_{xx}Q_j + (\beta_j^2 - \beta^2)Q_j = -\mu_0\omega^2\frac{N_{xx}}{N_x}\left[A|\mathbf{Q}|^2 Q_j + \frac{1}{2}B\mathbf{Q}^2 Q_j^*\right]. \tag{34}$$

The corresponding conservation laws are

$$\partial_t w = -\left(\partial_x S_x + \frac{\beta v}{k}\partial_z w\right), \tag{35}$$

$$F_x \equiv \partial_t g_x = \partial_x T_{xx} - \frac{\beta v}{k}\partial_z g_x, \tag{36}$$

$$F_z \equiv \partial_t g_z = -\frac{\beta}{\omega}\left(\partial_x S_x + \frac{\beta v}{k}\partial_z w\right) = \frac{\beta}{\omega}\partial_t w, \tag{37}$$

where w, $\mathbf{S}$, and $\mathbf{T}$ are

$$w = \epsilon|\mathbf{Q}|^2, \tag{38}$$

$$S_x = -\frac{i}{2\mu_0\omega}(\mathbf{Q}^*\cdot\partial_x\mathbf{Q} - \mathbf{Q}\cdot\partial_x\mathbf{Q}^*), \tag{39}$$

$$S_z = \frac{\beta}{\mu_0\omega}|\mathbf{Q}|^2 = \frac{\beta}{\mu_0\epsilon\omega}w, \tag{40}$$

$$T_{xx} = -\frac{1}{\mu_0\omega^2}|\partial_x\mathbf{Q}|^2 + \frac{1}{4\mu_0\omega^2}\partial_{xx}|\mathbf{Q}|^2 + \frac{N_{xx}}{4N_x}\left[A|\mathbf{Q}|^4 + \frac{1}{2}B|\mathbf{Q}^2|^2\right], \tag{41}$$

$$T_{xz} = T_{zx} = \frac{i\beta}{2\mu_0\omega^2}(\mathbf{Q}^*\cdot\partial_x\mathbf{Q} - \mathbf{Q}\cdot\partial_x\mathbf{Q}^*) = -\frac{\beta}{\omega}S_x = -\frac{\beta v}{k}g_x, \tag{42}$$

$$T_{zz} = -\frac{\beta^2}{\mu_0\omega^2}|\mathbf{Q}|^2 = -\frac{\beta^2}{k^2}w = -\frac{\beta v}{k}g_z. \tag{43}$$

It is clear that Eqs. (35) and (37) are the same. This means that the description of the stress is reduced to the component T_{xx} only.

From Eq. (35) it follows that the energy density varies in the course of space-time propagation in the z direction due to the transverse flow S_x (its derivation). The existence of this flow appears as the diffraction of the field. As follows from Eq. (36), the force F_x that acts in the transverse x direction during space-time propagation appears due to the transverse stress T_{xx} in the field. This interpretation can be reversed. The force that acts in the field during the space-time propagation gives rise to stress and causes the transverse flow of energy of the field. In other words, one can say that the effect of this force manifests itself as diffraction. This force is also nonzero in vacuum. Therefore the existence of this force can be seen as an intrinsic property of the field.

The given interpretation naturally stimulates the question of what the source of this force is. The discussion of this question is beyond the scope of this paper and therefore we shall not deal with it here.

From Eq. (36) it follows that there is no transverse force in the diffraction-free field, i.e., when $S_x = g_x = 0$. We shall study properties of such a field. To make our investigation more simple, in this article we shall confine ourselves to the cases that can be described by the scalar NSE [e.g., TE polarization ($E_y = 0$), TM polarization ($E_x = 0$), and circular polarization ($E_x = E_z e^{\pm i\pi/2}$)]. This means that $\mathbf{Q}$ is substituted by Q in Eqs. (34)–(43).

V. DIFFRACTION-FREE FIELD

To study problems of propagation it is suitable to introduce the transformation

$$x' = \beta x, \quad z' = \beta z, \quad \tau = \beta\left(z - \frac{\beta v}{k}t\right). \tag{44}$$

These variables are dimensionless. Using this transformation we obtain from Eq. (34)

$$i\partial_{z'}Q = -\frac{1}{2}\partial_{x'x'}Q - \kappa|Q|^2Q, \tag{45}$$

where $Q = Q(x', z', \tau)$ and

$$\kappa = \mu_0\left(\frac{vk}{\beta}\right)^2 \frac{N_{xx}}{2N_x}\left(A + \frac{1}{2}B\right). \tag{46}$$

Let us note that dimensionless time τ is only a parameter in this equation. To describe not only optical beams (stationary solutions) but also pulses we shall keep this dependence hereafter.

The corresponding laws of conservation have the form

$$\partial_{z'}w = -\partial_{x'}S, \tag{47}$$

$$F = \partial_{z'}S = \partial_{x'}T, \tag{48}$$

with

$$w = |Q|^2, \tag{49}$$

$$S = -\frac{i}{2}(Q^*\partial_{x'}Q - Q\partial_{x'}Q^*), \tag{50}$$

$$T = -|\partial_{x'}Q|^2 + \frac{1}{4}\partial_{x'x'}|Q|^2 + \frac{1}{2}\kappa|Q|^4. \tag{51}$$

These quantities are proportional to the original quantities. Hereafter we shall omit the dash over x and z. Our aim is to study the properties and behavior of a diffraction-free field. Such a field has to fulfill the condition

$$S(x, z, \tau) = 0, \tag{52}$$

if one uses the definition (2). The conservation laws for the diffraction-free field are

$$\partial_z w = 0, \tag{53}$$

$$\partial_x T = 0. \tag{54}$$

From these relations it follows that the energy density does not depend on z and the stress T does not depend on x.

We shall search for a general diffraction-free solution of Eq. (45). Let us introduce the substitution

$$Q = A\, e^{i\varphi}, \tag{55}$$

where A and φ are real quantities. From Eq. (45) we obtain

$$\partial_z A = -\partial_x A\partial_x\varphi - \frac{1}{2}\partial_{xx}\varphi, \tag{56}$$

$$A\partial_z\varphi = \frac{1}{2}\partial_{xx}A - \frac{1}{2}A(\partial_x\varphi)^2 + \kappa A^3 \tag{57}$$

and from (52) we have

$$S = A^2\partial_x\varphi = 0. \tag{58}$$

Ignoring the trivial solution we get

$$\partial_x\varphi = 0 \tag{59}$$

and

$$\partial_z A = 0, \tag{60}$$

$$A\partial_z\varphi = \frac{1}{2}\partial_{xx}A + \kappa A^3. \tag{61}$$

Equation (60) corresponds to Eq. (53). Since φ does not depend on x and A does not depend on z, Eq. (61) can be split into

$$\partial_z\varphi = \frac{1}{2}\Delta\beta, \tag{62}$$

$$\partial_{xx}A - \Delta\beta A + 2\kappa A^3 = 0, \tag{63}$$

where $\Delta\beta = \Delta\beta(\tau)$. The solution of Eq. (62) is $\varphi(z,\tau) = \frac{1}{2}\Delta\beta(\tau)(z - z_0) + C_1$, where z_0 and C_1 are integration constants. Without loss of generality we can set $C_1 = 0$. It is obvious that the constant $\Delta\beta$ represents the nonlinear phase correction of the linear constant of propagation.

For the stress T of the field we have

$$T = -\frac{1}{2}\left[(\partial_x A)^2 - A\partial_{xx}A - \kappa A^4\right] \tag{64}$$

$$= -\frac{1}{2}\left[(\partial_x A)^2 - \Delta\beta A^2 + \kappa A^4\right], \tag{65}$$

where Eq. (63) has been used. Taking into account Eqs. (54) (T cannot depend on x) and (60) (A cannot depend on z), it is obvious that T is a function of τ only. Apart from the value of κ (if it is positive or negative), we can summarize properties of the diffraction-free field as follows: (i) the transversal distribution of the amplitude cannot change during the propagation; (ii) the transversal distribution of the phase is uniform during the propagation; and (iii) the stress in the diffraction-free field is constant or, in other words, the force does not act in the diffraction-free field.

To gain more information about properties of the diffraction-free field we shall analyze properties of the diffraction-free field in relation to the magnitude of T and $\Delta\beta$ and solve Eq. (63). Since the stress T of a diffraction-free field is constant, each corresponding solution has to lie on the isohypse of a surface $T = T(A, \partial_x A)$. The isohypses that are closed lines correspond to periodic solutions. These solutions are the subject of our interest.

A. Case $\kappa > 0$ (focusing case)

1. $\Delta\beta > 0$

The surface T is shown in Fig. 1. The surface has two maxima at the points $(-\sqrt{\Delta\beta/2\kappa}, 0)$ and $(0, \sqrt{\Delta\beta/2\kappa})$.

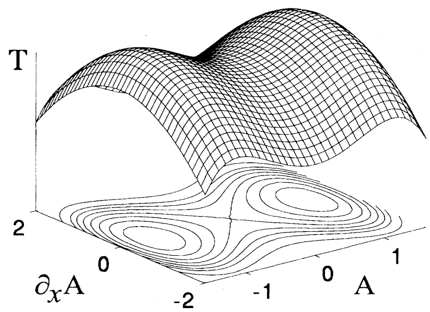

FIG. 1. Behavior of the stress T (in arbitrary units) as a function of A and $\partial_x A$ is demonstrated for the case $\kappa > 0$ and $\Delta\beta > 0$. The contour map corresponds to isohypses on the T surface.

All values of the stress fulfilling the condition

$$T \leq T_{\max} \equiv \frac{\Delta\beta^2}{8\kappa} \tag{66}$$

are acceptable. The corresponding solutions are periodic functions as follows from the shape of the contours.

2. $\Delta\beta < 0$

The surface T is shown in Fig. 2. The surface has one maximum at the point $(0,0)$. The stress has to be negative, i.e.,

$$T \leq T_{\max} \equiv 0. \tag{67}$$

The corresponding solutions are again periodic functions as follows from the shape of the contours.

All diffraction-free solutions for the case $\kappa > 0$ are

$$A(x) = \pm A_0 \,\mathrm{dn}[A_0\sqrt{\kappa}(x - x_0), m], \tag{68}$$

for

$$T \in \left\langle 0, \frac{\Delta\beta^2}{8\kappa} \right\rangle, \quad \Delta\beta > 0 \tag{69}$$

and

$$A(x) = \pm A_0 m_1 \,\mathrm{cn}[A_0\sqrt{\kappa}(x - x_0), m_1] \tag{70}$$

for

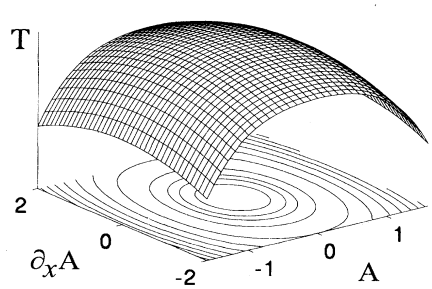

FIG. 2. Same as in Fig. 1 but for $\Delta\beta < 0$.

$$T \in \langle -\infty, 0\rangle, \quad \Delta\beta \text{ arbitrary}, \tag{71}$$

where A_0 is a constant, $\mathrm{cn}(y,m)$ and $\mathrm{dn}(y,m)$ are the Jacobian elliptic functions, and $m \in \langle 0,1\rangle$ is their modulus. The stress T and the correction of the propagation constant $\Delta\beta$ can be expressed by means of A_0 and m as

$$T = \frac{1}{2}\kappa A_0^4(1 - m^2), \tag{72}$$

$$\Delta\beta = \kappa A_0^2(2 - m^2) \tag{73}$$

for solution (68) and

$$T = \frac{1}{2}\kappa A_0^4 m_1(m_1^2 - 1), \tag{74}$$

$$\Delta\beta = \kappa A_0^2(2m_1^2 - 1) \tag{75}$$

for solution (70).

Let us note that from a practical point of view it is more suitable to use parameters A_0 and m for the determination of a solution. These parameters can be functions of τ.

The solutions are the periodic functions with a period

$$L_x(m) = \frac{2\mathbf{K}(m)}{\sqrt{\kappa}A_0} \tag{76}$$

for solution (68) and

$$L_x(m_1) = \frac{4\mathbf{K}(m_1)}{\sqrt{\kappa}A_0} \tag{77}$$

for solution (70), where $\mathbf{K}(m)$ is a complete elliptic integral of the first kind.

The energy density w is also a periodic function with the period (76). The total energy corresponding to this period is

$$W(m) = \int_{-L_x(m)/2}^{L_x(m)/2} w(x,m)\,dx = 2\frac{A_0}{\sqrt{\kappa}}\mathbf{E}(m) \tag{78}$$

for (68) and

$$W(m_1) = \frac{2A_0}{\sqrt{\kappa}}\left[\mathbf{E}(m_1) - (1 - m_1^2)\mathbf{K}(m_1)\right] \tag{79}$$

for (70), where $\mathbf{E}(m)$ is a complete elliptic integral of the second kind.

Let us take notice of particular cases. If $m = 0$ we have for solution (68)

$$A(x) = A_0, \tag{80}$$

$$\Delta\beta = 2\kappa A_0^2, \tag{81}$$

$$T = \frac{1}{2}\kappa A_0^4, \tag{82}$$

$$L_x = \frac{\pi}{\sqrt{\kappa}A_0}, \tag{83}$$

$$W = \frac{\pi A_0}{\sqrt{\kappa}}. \tag{84}$$

The solution has the shape of a plane wave, where the influence of nonlinearity manifests itself in the phase only. This solution is entirely phase modulated.

L_x is a fictive period that appears as the limit. But we know from the stability analysis that the plane wave is unstable with respect to perturbations that have a period longer than L_x. It is also worthwhile to remark that there is nonzero stress in a plane wave.

If $m = 1$ we have for solution (68)

$$A(x) = A_0 \operatorname{sech}[A_0\sqrt{\kappa}(x - x_0)], \quad (85)$$
$$\Delta\beta = \kappa A_0^2, \quad (86)$$
$$T = 0, \quad (87)$$
$$L_x = \infty, \quad (88)$$
$$W = \frac{2A_0}{\sqrt{\kappa}}. \quad (89)$$

The solution has the form of the spatial soliton. The influence on the phase is, in this case, one-half of that in the case of the plane wave. The distribution of the energy density is not periodic and the total energy is that less the energy corresponding to one period in the plane wave. But the most pronounced features of the spatial soliton is the fact that the stress is zero. For solution (70) and $m_1 = 1$ the results are the same. In the case $m_1 = 0$ we have a trivial solution.

If $m_1 = 1/\sqrt{2}$ then $\Delta\beta = 0$ and the phase of the solution is not affected. This solution is entirely amplitude modulated.

B. Case $\kappa < 0$ (defocusing case)

1. $\Delta\beta > 0$

The surface T is shown in Fig. 3. The surface T is saddle shaped and periodic solutions do not exist.

2. $\Delta\beta < 0$

The surface T is shown in Fig. 4. Periodic solutions exist if the stress lies in the interval

$$T \in \left\langle -\frac{\Delta\beta^2}{8|\kappa|}, 0 \right\rangle. \quad (90)$$

The solution is

$$A(x) = \pm A_0 m_2 \operatorname{sn}[A_0\sqrt{|\kappa|}(x - x_0), m_2], \quad (91)$$

where $\operatorname{sn}(y, m)$ is the Jacobian elliptic function. For T and $\Delta\beta$ we have

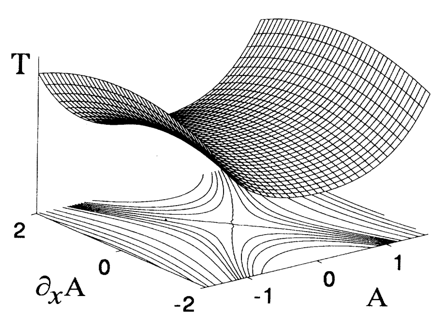

FIG. 3. Same as in Fig. 1 but for $\kappa < 0$.

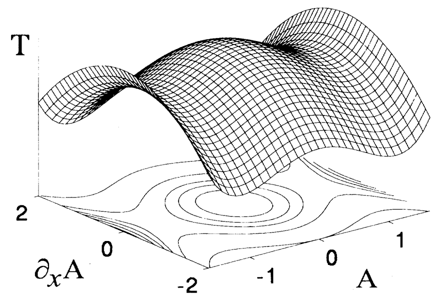

FIG. 4. Same as in Fig. 3 but for $\Delta\beta < 0$.

$$T = -\frac{1}{2}|\kappa| A_0^4 m_2^2, \quad (92)$$
$$\Delta\beta = -|\kappa| A_0^2 (1 + m_2^2). \quad (93)$$

The period of the solution is

$$L_x(m_2) = \frac{4\mathbf{K}(m_1)}{\sqrt{\kappa} A_0} \quad (94)$$

and the total energy corresponding to half of this period is

$$W(m_2) = \frac{2A_0}{\sqrt{|\kappa|}} \left[\mathbf{K}(m_2) - \mathbf{E}(m_2)\right]. \quad (95)$$

Let us take notice of particular cases. If $m_2 = 0$ then $A(x) = 0$, i.e., a trivial solution. If $m_2 = 1$ then

$$A(x) = \pm A_0 \tanh[A_0\sqrt{|\kappa|}(x - x_0)], \quad (96)$$
$$\Delta\beta = -2|\kappa| A_0^2, \quad (97)$$
$$T = -\frac{1}{2}|\kappa| A_0^4, \quad (98)$$
$$L_x = \infty, \quad (99)$$
$$W = \infty. \quad (100)$$

This solution is known as a dark spatial soliton. Its nonlinear phase and the stress are the same as those for the plane wave in the focusing case, but with opposite sign.

It is known about the dark soliton that it is stable with respect to arbitrary disturbance. Furthermore, the plane wave in the defocusing case is resistant to arbitrary disturbance. Let us note that one can obtain a plane wave from the expression for the dark soliton limiting $x_0 \to \pm\infty$.

VI. VALIDITY OF THE PARAXIAL APPROXIMATION

Our solutions describe the physical situation investigated under the assumption that the paraxial condition (33) is fulfilled. Substituting any of our solutions into (33) we arrive at

$$\frac{1}{2}|\Delta\beta| \ll 1. \quad (101)$$

Since $\Delta\beta$ is the contribution of the nonlinear interaction to the linear constant of propagation, we can say that the paraxial approximation is valid if the nonlinear interaction does not influence the propagation constant too much.

There is one special case that we would like to point out. Substituting into (33) any of our solutions corresponding to the plane wave, we arrive at the condition

$$|\kappa|A_0^2 \ll 1. \tag{102}$$

This means that the paraxial approximation is not fulfilled by itself for an arbitrary plane wave, as it is sometimes supposed. The maximum power of the plane wave is limited in the paraxial approximation.

VII. CONCLUSION

This paper contains several results of our investigation of the properties of diffraction-free fields. The derivation of the laws of conservation corresponding to the vector NSE describing the propagation of an electric field in the nonlinear Kerr medium are presented. These laws of conservation are derived for a (3+1)D medium (a bulk material) and as well for a (2+1)D medium (a nonlinear planar waveguide). Their physical interpretation is given. It seems to us that the interpretation presented can be a base for an alternative view on the behaviour of the electromagnetic field.

The formulation involves arbitrary polarization. It is worthwhile to point out that the vector formulation is necessary if one wants to investigate, e.g., the interaction of two spatial optical solitons colliding under a general crossing angle between them, although the propagation of each of them separately can be described by a scalar NSE.

The condition for the diffraction-free field, which is stated at the beginning of the article, is then used to obtain a solution of the NSE in the scalar case. Although the solutions obtained are known, different physical characteristics (the stress of field and the force) are introduced. From this point of view there are no forces in the diffraction-free fields and the stress of these fields is constant.

It is again necessary to remember that our results are not valid for the stationary optical beams only, but also for the optical pulses (the quasimonochromatic approximation limits the length of the pulses). If, for example, $A_0 = A_0\ \mathrm{sech}(\tau)$ and $m = 1$, then relation (85) describes the spatial soliton in the form of the pulse. The amplitude distribution of such a pulse spatial soliton is demonstrated in Fig. 5.

Not only A_0 but also the modulus m (or the stress T or/and the nonlinear phase correction $\Delta\beta$) can depend on τ. In this case the character of the field can change during a pulse and the behavior is more complicated. Finally, it is shown that a strong plane wave does always not fulfill the paraxial approximation, as is sometimes expected.

In conclusion it is worthwhile to remember that if the function $Q(x,z)$ is the solution of NSE, then the function

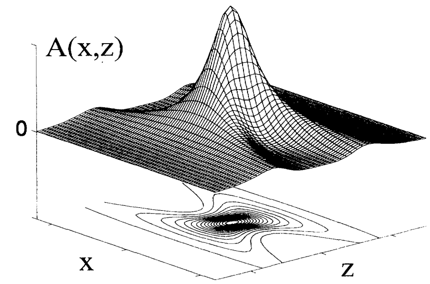

FIG. 5. Space distribution of the amplitude of the pulse spatial soliton for an arbitrary time moment if the amplitude A_0 has the form $A_0 = A_0\ \mathrm{sech}(\tau)$.

$$U(x,z) = C\ Q(C(x-x_0), C^2(z-z_0))\ e^{i\alpha},$$

where C, x_0, z_0, and α are real constants, is also the solution of the NSE. This transformation can sometimes be useful if one wants to rewrite the solution to a more suitable form.

APPENDIX

Using the quasimonochromatic approximation we shall show a relation between the density of flow defined on the basis of the Maxwell equations (Poynting vector) and the density of flow defined on the basis of the wave equations. The complex electric vector $\hat{\mathbf{E}}$ and the complex magnetic vector $\hat{\mathbf{H}}$ fulfill the Maxwell equations

$$\nabla \times \hat{\mathbf{H}} - \epsilon\partial_t\hat{\mathbf{E}} = \mathbf{0}, \tag{A1}$$

$$\nabla \times \hat{\mathbf{E}} + \mu\partial_t\hat{\mathbf{H}} = \mathbf{0}, \tag{A2}$$

$$\nabla \cdot \hat{\mathbf{H}} = 0, \tag{A3}$$

$$\nabla \cdot \hat{\mathbf{E}} = 0. \tag{A4}$$

The conservation law of energy

$$\nabla \cdot \mathbf{S} + \partial_t w = 0 \tag{A5}$$

can be derived from the Maxwell equations, where

$$w = \epsilon|\hat{\mathbf{E}}|^2 + \mu_0|\hat{\mathbf{H}}|^2, \tag{A6}$$

$$\mathbf{S} = \hat{\mathbf{E}}^* \times \hat{\mathbf{H}} + \hat{\mathbf{E}} \times \hat{\mathbf{H}}^*, \tag{A7}$$

where w and $\mathbf{S}$ are the energy density and the Poynting vector, respectively.

Starting from Eqs. (A1)–(A4), one can derive wave equations

$$\nabla^2\hat{\mathbf{E}} - \mu\epsilon\partial_{tt}\hat{\mathbf{E}} = \mathbf{0}, \tag{A8}$$

$$\nabla^2\hat{\mathbf{H}} - \mu\epsilon\partial_{tt}\hat{\mathbf{H}} = \mathbf{0}. \tag{A9}$$

From these wave equations we get other conservation laws

$$\partial_j(\hat{\mathbf{E}}^* \cdot \partial_j\hat{\mathbf{E}} - \hat{\mathbf{E}} \cdot \partial_j\hat{\mathbf{E}}^*) - \mu\epsilon\partial_t(\hat{\mathbf{E}}^* \cdot \partial_t\hat{\mathbf{E}} - \hat{\mathbf{E}} \cdot \partial_t\hat{\mathbf{E}}^*) = 0, \tag{A10}$$

$$\partial_j(\hat{\mathbf{H}}^* \cdot \partial_j \hat{\mathbf{H}} - \hat{\mathbf{H}} \cdot \partial_j \hat{\mathbf{H}}^*)$$

$$-\mu\epsilon\partial_t(\hat{\mathbf{H}}^* \cdot \partial_t \hat{\mathbf{H}} - \hat{\mathbf{H}} \cdot \partial_t \hat{\mathbf{H}}^*) = 0. \quad \text{(A11)}$$

Using expression (5) for the vectors $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ and the quasimonochromatic approximation (6), the conservation laws (A10) and (A11) are

$$\nabla \cdot \mathbf{S}_E + \partial_t w_E = 0, \quad \text{(A12)}$$

$$\nabla \cdot \mathbf{S}_H + \partial_t w_H = 0, \quad \text{(A13)}$$

where

$$w_E = \epsilon |\mathbf{E}|^2, \quad \text{(A14)}$$

$$w_H = \mu_0 |\mathbf{H}|^2, \quad \text{(A15)}$$

$$S_{Ej} = -\frac{i}{2\omega\mu_0}(\mathbf{E}^* \cdot \partial_j \mathbf{E} - \mathbf{E} \cdot \partial_j \mathbf{E}^*), \quad \text{(A16)}$$

$$S_{Hj} = -\frac{i}{2\omega\epsilon}(\mathbf{H}^* \cdot \partial_j \mathbf{H} - \mathbf{H} \cdot \partial_j \mathbf{H}^*), \quad \text{(A17)}$$

where $j = x, y, z$. The law of conservation (A5) is formally the same.

It is evident that w_E (w_H) is the density of the electric (magnetic) energy of the electromagnetic field. Comparing Eqs. (A5), and (A12) and (A13), one can see that $\mathbf{S}_E$ ($\mathbf{S}_H$) has the same meaning for the the electric (magnetic) field as the Poynting vector for the electromagnetic field. Therefore we can say that the Poynting vector $\mathbf{S}$ can be decomposed to separate contributions of electric and magnetic fields in the quasimonochromatic approximation.

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